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THE TEMPERATURE DISTRIBUTION IN ROTATING THICK-WALLED CYLINDERS HEATED BY RADIATION

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Abstract—Analytical solutions are obtained for the temperature distribution in circular cylinders heated by radiation from a distant source. The cylinder is assumed to be rotating with respect to the source, about its geometrical axis, and to be inclined at a given constant angle to the radiation direction. Quasi-steady temperatures are found as series expressions in terms of orthogonal functions for the solid cylinder, and for the hollow cylinder with an adiabatic hole. The resulting temperature distributions are examined to ascertain the significance of the rotation, radial temperature gradients and the wall thickness of the body.

NOMENCLATURE

a, absorptivity of surface exposed to radiation;
a_n, b_n, integration constants in equation (27);
A_n, B_n, real series coefficients for temperature;
C_n, complex series coefficients for hollow cylinder temperatures;
$C_n^{(s)}$, complex series coefficients for solid cylinder temperatures;
D_n, $D_n^{(2)}$, non variable part of temperature series coefficients;
G_n, series coefficients for radiation input function;
k, thermal conductivity [Btu ft^{-1} h^{-1} R^{-1}];
K_s, density of solar radiation [Btu ft^{-2} h^{-1}];
r, radius coordinate [ft];
r_i, r_o, inside and outside radial dimensions [ft];
s, wall thickness [ft];
t, t', time [h];
T, hollow cylinder temperature [°R];
$T^{(s)}$, solid cylinder temperature [°R];
T_c, datum for temperature [°R];
$\hat{T}$, temperature variable in equation [11];
$\tilde{T}$, temperature variable in equation [13];
T_0, constant temperature in equation [11];
w, $i^{\frac{3}{2}}n^{\frac{1}{2}}\rho$;
x, y, $\tilde{z}$, Cartesian coordinates;
z, complex variable, $re^{i\phi}$.

Greek symbols
α, thermal diffusivity [ft^{2}h^{-1}];
β, $(\omega r_o^2/\alpha)^{\frac{1}{2}}$;
β_i, $(\omega r_i^2/\alpha)^{\frac{1}{2}}$;
γ, $r_o a K_s \sin\psi/(kT_0)$;
ε, emissivity;
ζ, variable in change of variable equation (25);
$\bar{\eta}$, Euler's constant;
θ, angular coordinate fixed in the cylinder;
λ, $4r_o\sigma\varepsilon T_0^3/k$;
ξ, nondimensional radius, r/r_o;
ρ, $(\omega r^2/\alpha)^{\frac{1}{2}}$;
ρ_i, $(\omega r_i^2/\alpha)^{\frac{1}{2}}$;
σ, Stefan–Boltzmann constant [0·171 × 10^{-8} Btu ft^{-2}h^{-1}R^{-1}];
τ, nonlinear temperature [°R];

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ϕ, angular coordinate fixed in space [rad];
ψ, inclination of cylinder axis to radiation direction;
ω, speed of angular rotation [rad h^{-1}].

1. INTRODUCTION

A BODY traveling through interplanetary space will attain some temperature distribution determined by the nonuniformity of solar illumination of the surface and by the rates of absorption and emission of thermal radiation. In recent years there have appeared in the literature papers concerned with heat conduction in various types of bodies with a condition of radiant energy interchange prescribed on the surface. In general, these papers consider the body to be placed in a vacuum, to receive radiant energy from a distant source, to reradiate energy to a heat sink at absolute zero and to absorb and emit radiation as gray bodies. Although the motivation for this work has been the thermal control of Earth satellites and space vehicles, the results of the analyses can be applicable under certain conditions to other situations involving the radiation heating of bodies.

The problem of the solar heating of a rotating thin cylindrical shell has been considered by Charnes and Raynor in [1]. Nichols [2] has obtained results for thin-walled bodies with the geometry of spheres, cones and cylinder, including the effects of rotation and internal reradiation for the hollow sphere. Roberts [3] has obtained approximate formulas for the temperatures in solid cylinders but does not consider the possibility of rotation. Jenness [14] treats the solid cylinder with a uniformly varying surface temperature but does not consider the role of rotation. Hrycak and Helgans [4] have investigated the thin-walled cylindrical shell and accounted for the more general problem of internal reradiation.

In a closely related paper, Olmstead and Raynor [5] depart from the assumption of thin walls and consider the problem of the rotating solid cylinder. In this investigation it was determined that the magnitude of the corresponding temperature variations in the solid body were smaller than in the thin-walled cylinder for all angular velocities and differed in one numerical example by almost two orders of magnitude, indicating a need for an investigation of the limits of usefulness of the thin-walled assumption. Nermann [6] has undertaken the analysis of the cylinder problem with the inclusion of the necessary radial conduction terms in the governing differential equation in order that the resulting solution be valid for the hollow cylinder configuration. However, in this latter work, as well as in [5], the solution has been examined only for the limiting cases of either very slow or very fast rotational speeds.

In this paper a cylinder of arbitrary wall thickness is considered to be rotating with constant angular velocity about its geometric axis, the axis being inclined at a given angle to the direction of the incoming radiation. Thermal radiation from a single distant source is heating the body and thermal equilibrium is attained by means of reradiation to the surrounding space. Conduction and convection effects with the surroundings are neglected at the surface of the body. Internal reradiation is neglected on the basis of the conclusion reached in [2] that the body rotation influences the temperatures to a much greater degree than internal radiation. The directional dependence of absorptivity and the dependence of absorptivity and emissivity upon temperature and wavelength are neglected. The thermodynamic equilibrium temperature distribution obtained in the body is quasi-steady, varying periodically at a point in the body but constant in a fixed frame of reference. Longitudinal conduction is neglected; the heat flow is radial and circumferential and the resulting temperature distribution two dimensional.

2. FORMULATION OF THE PROBLEM

The cylindrical coordinate system (r, θ, $\tilde{z}$)

fixed in the body is shown in Fig. 1, with θ measured in a sense opposite to the rotation, ω. The axis of symmetry is inclined at angle ψ to the direction of the parallel rays of incoming radiation. Taking diffusivity α as constant, the governing partial differential equation can be written

$$\frac{\partial^2 \tau}{\partial r^2} + \frac{1}{r}\frac{\partial \tau}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tau}{\partial \theta^2} = \frac{1}{\alpha}\frac{\partial \tau}{\partial t}. \tag{1}$$

A transformation that has been useful in solving problems with moving heat sources [7] eliminates the explicit dependence of the temperature

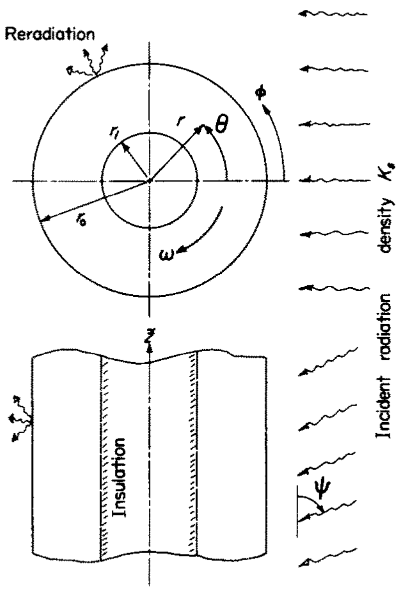

FIG. 1. The rotating coordinate system in the cylinder. Position shown is for $t = 0$; $\phi = \theta - \omega t$.

on time. For polar coordinates this transformation is

$$\varphi = \theta - \omega t \qquad t' = t. \tag{2}$$

Fixed in a stationary frame of reference, the temperature becomes the quasi-steady temperature,

$$T(r, \varphi) = \tau(r, \theta, t) \tag{3}$$

characterized by

$$\frac{\partial T}{\partial t'} = 0. \tag{4}$$

Using equations (2–4), equation (1) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\omega}{\alpha}\frac{\partial T}{\partial \varphi} = 0. \tag{5}$$

The boundary conditions for the problem involve prescribed surface values for the heat flux. On the external surface the body is heated by the absorption of radiant energy from the distant source and suffers an energy loss by reradiation to space. The gradient of temperature is determined by the net rate of heat conduction per unit area with the local absorbed radiation prescribed by Lambert's cosine law and the reradiation governed by the Stefan–Boltzmann law. There results

$$k\frac{\partial \tau}{\partial r} = aK_s \sin\psi \cos^+(\theta - \omega t) - \sigma\varepsilon\tau^4 \qquad \text{on } r = r_o \tag{6}$$

where a is the average absorptivity of the cylinder surface, K_s is the radiant energy flux in a plane normal to the direction of parallel radiation, σ is the Stefan–Boltzmann constant and ε is the average emissivity of the cylinder surface. The rectified cosine function is defined

$$\cos^+(\theta - \omega t) = \begin{cases} \cos(\theta - \omega t) & -\frac{\pi}{2} \leqslant \theta - \omega t \leqslant \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leqslant \theta - \omega t \leqslant \frac{3\pi}{2}. \end{cases} \tag{7}$$

The adiabatic boundary condition for the inside is

$$\frac{\partial \tau}{\partial r} = 0 \qquad \text{on} \quad r = r_i. \tag{8}$$

When the transformation equations (2) and (3) are introduced into the boundary conditions, equations (6) and (8) become

$$k\frac{\partial T}{\partial r} = aK_s \sin\psi \cos^+\varphi - \sigma\varepsilon T^4 \qquad \text{on} \quad r = r_o \tag{9}$$

and

$$\frac{\partial T}{\partial r} = 0 \qquad \text{on} \quad r = r_i. \tag{10}$$

In order to linearize the reradiation heat flux term in equation (9), it is defined that

$$T = T_0(1 + \hat{T}). \tag{11}$$

It is assumed that $\hat{T}$ is small compared to unity, and this smallness is utilized to make the approximation

$$T^4 \cong T_0^4(1 + 4\hat{T}). \tag{12}$$

Then it becomes possible to eliminate the fourth power of temperature in the outside boundary condition by the substitution of

$$\tilde{T} = \hat{T} + \tfrac{1}{4} \tag{13}$$

in terms of which

$$T^4 = 4T_0^4\tilde{T}. \tag{14}$$

In addition the following nondimensional quantities are introduced;

$$\xi = r/r_o \tag{15}$$

$$\beta = (\omega r_o^2/\alpha)^{\frac{1}{2}} \tag{16}$$

$$\gamma = r_o a K_s \sin\psi/(kT_0) \tag{17}$$

$$\lambda = 4r_o\sigma\varepsilon T_0^3/k. \tag{18}$$

Substituting equations (13) through (16), the differential equation (5) becomes

$$\frac{\partial^2\tilde{T}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\tilde{T}}{\partial\xi} + \frac{1}{\xi^2}\frac{\partial^2\tilde{T}}{\partial\varphi^2} + \beta^2\frac{\partial\tilde{T}}{\partial\varphi} = 0 \tag{19}$$

and with equations (17) through (18) the boundary conditions assume the forms

$$\frac{\partial\tilde{T}}{\partial\xi} = \gamma\cos^+\varphi - \lambda\tilde{T} \qquad \text{on} \quad r = r_o \tag{20}$$

and

$$\frac{\partial\tilde{T}}{\partial\xi} = 0 \qquad \text{on} \quad r = r_i. \tag{21}$$

3. SOLUTION OF THE PROBLEM FOR THE TEMPERATURES IN THE HOLLOW CYLINDER

To reduce the governing partial differential equation to an ordinary differential equation, consider the temperature to be expanded in a Fourier series of complex form;

$$\tilde{T}(r, \varphi) = \sum_{n=-\infty}^{\infty} C_n(r)\exp(-in\varphi) \tag{22}$$

where the coefficients are given by

$$C_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{T}(r, \varphi)\exp(in\varphi)\,\mathrm{d}\varphi. \tag{23}$$

This expansion requires that the temperature function satisfy the Dirichlet conditions and be periodic, requirements which physical intuition indicates are readily met.

The following ordinary differential equation is obtained:

$$\frac{\mathrm{d}^2C_n}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}C_n}{\mathrm{d}\xi} - \frac{n^2}{\xi^2}C_n - in\beta^2C_n = 0. \tag{24}$$

By making the change of variable

$$\zeta^2 = -in\beta^2\xi^2, \tag{25}$$

equation (24) assumes the usual form of Bessel's differential equation,

$$\zeta^2\frac{\mathrm{d}^2C_n}{\mathrm{d}\zeta^2} + \zeta\frac{\mathrm{d}C_n}{\mathrm{d}\zeta} + (\zeta^2 - n^2)C_n = 0 \tag{26}$$

for which the solution is known to be

$$C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) = a_nJ_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) + b_nY_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho). \tag{27}$$

$J_n(z)$ and $Y_n(z)$ are Bessel functions of the first and second kinds, respectively; a_n and b_n are constants of integration and ρ is defined by $\rho = \beta r_o^{-1}r$. Examination of the respective power series for the Bessel functions [8], determines that the C_0 coefficient is a constant. Then equation (22) is written

$$\tilde{T}(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho, \varphi) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho)\exp(-in\varphi). \tag{28}$$

For the inside boundary condition to be satisfied one has

$$\frac{\mathrm{d}C_n}{\mathrm{d}\xi} = a_ni^{\frac{3}{2}}n^{\frac{1}{2}}\beta J_n'(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) + b_ni^{\frac{3}{2}}n^{\frac{1}{2}}\beta Y_n'(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho)$$

$$= 0 \qquad \text{on} \quad r = r_i \tag{29}$$

where the prime denotes differentiation with respect to the function argument. Designating the radius variable ρ for $r = r_i$ as β_i and using equation (29) in (27) one obtains

$$C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) = -b_n \frac{Y_n'(i^{\frac{3}{2}}n^{\frac{1}{2}}\beta_i)}{J_n'(i^{\frac{3}{2}}n^{\frac{1}{2}}\beta_i)} J_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) + b_n Y_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho). \qquad (30)$$

The outside boundary condition determines the remaining constant of integration. First one must consider the expanded form of the radiation input function. It is found by the usual method that

$$\cos^+ \varphi = \frac{1}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G_n \exp(-in\varphi) \qquad (31)$$

where

$$G_n = \begin{cases} \frac{1}{4} & |n| = 1 \\ \dfrac{(-1)^{(n+2)/2}}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

In terms of the Fourier coefficients, equation (20) becomes

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mathrm{d}C_n}{\mathrm{d}\xi} \exp(-in\varphi) = \frac{\gamma}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \gamma G_n \exp(-in\varphi) - \lambda C_0 - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \lambda C_n \exp(-in\varphi) \quad \text{on} \quad r = r_o. \qquad (32)$$

This equation independently requires that

$$\gamma = \pi\lambda C_0 \qquad (33)$$

and

$$\frac{\mathrm{d}C_n}{\mathrm{d}\xi} = \gamma G_n - \lambda C_n \quad \text{on} \quad r = r_o. \qquad (34)$$

Because T_0 is as yet unspecified, equation (33) does not serve to determine C_0. To choose T_0 in a manner which minimizes the error of linearization in equation (11), it is determined that T_0 be made the mean value of cylinder temperature, defined accordingly as

$$T_0 = \frac{1}{\pi(r_o^2 - r_i^2)} \int_{r_i}^{r_o} \int_0^{2\pi} T(r, \varphi) r \, \mathrm{d}\varphi \, \mathrm{d}r. \qquad (35)$$

Substituting

$$T(r, \varphi) = T_0[\tfrac{3}{4} + \tilde{T}(r, \varphi)] \qquad (36)$$

and equation (28), this relationship becomes

$$\frac{1}{\pi(r_o^2 - r_i^2)} \int_{r_i}^{r_o} \int_0^{2\pi} \Big[C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) \exp(-in\varphi)\Big] r \, \mathrm{d}\varphi \, \mathrm{d}r = \tfrac{1}{4} \qquad (37)$$

from which is obtained the result that

$$C_0 = \tfrac{1}{4}. \qquad (38)$$

Returning to equations (17) and (18), it is found that the mean cylinder temperature is given by the formula

$$T_0 = \left(\frac{aK_s \sin\varphi}{\pi\sigma\varepsilon}\right)^{\frac{1}{4}} \qquad (39)$$

and that the expressions for γ and λ become

$$\gamma = \frac{r_o}{k}(\pi\sigma\varepsilon a^3 K_s^3)^{\frac{1}{4}}, \qquad \lambda = \frac{4}{\pi}\gamma. \qquad (40)$$

Only b_n remains to be determined and it is found from equation (34) with the substitution of equation (30), yielding

$$b_n = \frac{\gamma G_n J_n'(w_i)}{Y_n'(w_i)[-w_o J_n'(w_o) - \lambda J_n(w_o)] + J_n'(w_i)[w_o Y_n'(w_o) + \lambda Y_n(w_o)]}. \qquad (41)$$

in terms of the definitions

$$w = i^{\frac{3}{2}}n^{\frac{1}{2}}\rho \qquad w_i = i^{\frac{3}{2}}n^{\frac{1}{2}}\beta_i \qquad w_o = i^{\frac{3}{2}}n^{\frac{1}{2}}\beta. \qquad (42)$$

The Fourier coefficients for the temperature distribution then assume their final form as

$$C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) = \frac{\gamma G_n[Y_n'(w_i)\,J_n(w) - J_n'(w_i)\,Y_n(w)]}{Y_n'(w_i)\,[w_o J_n'(w_o) + \lambda J_n(w_o)] - J_n'(w_i)\,[w_o Y_n'(w_o) + \lambda Y_n(w_o)]}. \tag{43}$$

The actual temperature values from equations (28) and (36) are given

$$T(r, \varphi) = T_0 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho)\exp(-in\varphi) + T_0. \tag{44}$$

4. TEMPERATURES IN THE SOLID CYLINDER

The corresponding solution for the radiation heating of the solid cylinder is treated by replacing the former internal boundary condition prescribing the gradient by the requirement that the temperature at the origin be finite. As a consequence the solid cylinder requirement of finite temperature for $r = 0$ leads to the conclusion that b_n in equation (27) must be zero. The other integration constant is determined from equation (34) so that for the solid cylinder the Fourier coefficients become

$$C_n^{(s)}(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho) = \frac{\gamma G_n J_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\rho)}{i^{\frac{3}{2}}n^{\frac{1}{2}}\beta J_n'(i^{\frac{3}{2}}n^{\frac{1}{2}}\beta) + \lambda J_n(i^{\frac{3}{2}}n^{\frac{1}{2}}\beta)}. \tag{45}$$

The temperatures in the solid cylinder for the case of no rotation are determined by considering the behavior of the series coefficients as ω, or equivalently ρ, approaches zero. It is appropriate to consider the small argument approximation for the Bessel function:

$$J_n(z) \cong z^n/(2^n n!) \quad (z \to 0). \tag{46}$$

The derivative in the expression for the coefficients is replaced by the identity

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z). \tag{47}$$

Substituting equations (46) and (47) into equation (45) and allowing ω to approach zero, one obtains

$$C_n^{(s)}(r) = \frac{\gamma G_n (r/r_o)^n}{\lambda + 2n}. \tag{48}$$

It is noted that in equation (48) the complex series coefficient is a real quantity and it follows that the temperature is symmetrical in the body about $\phi = 0$. This result is similar to that of the hollow cylinder for $\omega = 0$ in that the series simply consists of terms which are powers of the radius.

5. EVALUATION OF THE SOLUTIONS

The initial step in the evaluation of the solutions consists in replacing the Bessel function derivatives with different orders of the same function according to the identities [9]:

$$\begin{aligned} 2y_p'(z) &= y_{p-1}(z) - y_{p+1}(z) \\ zy_p'(z) &= py_p(z) - zy_{p+1}(z). \end{aligned} \tag{49}$$

Although the real and imaginary parts of J_n and Y_n can be resolved into Kelvin functions, these latter functions have only been tabulated for orders zero and one in the case of [10] and are tabulated by [11] for argument values limited to $|z| \leqslant 10$ that require tedious interpolation. Because a given nth term in the Fourier series solution involves functions of orders n, $n + 1$ and $n - 1$, it is necessary to compute functions of order $y_0(z)$, $y_1(z)$, . . . , $y_{N+1}(z)$ where N is the index value of the highest order term appearing in the suitably truncated series. In evaluating the general solution it is difficult to justify a limitation of the argument range because the arguments are functions not only of index n, but also of thermal diffusivity, radius and speed of angular rotation. Therefore it is necessary to the calculation of a single value of temperature to determine Bessel–Kelvin functions of many orders and for a like number of argument values. This necessary generation of sets of values of these functions is a computational problem well-suited to certain high speed digital computer techniques.

In calculating $J_n(z)$ and $Y_n(z)$ the power series representation is useful only for values of $|z|$ which are small. Asymptotic expansions of J_n and Y_n are useful only when $|z|$ is large with respect to n. For those cases where both n and $|z|$ are large, a method due to Goldstein and Thaler [12] is used. To initiate this recurrence technique the variable m is chosen larger than the greater of n and $|z|$ and it is assumed that $\bar{J}_{m+1}(z) = 0$ and $\bar{J}_m(z) = \bar{\varepsilon}$, where $\bar{\varepsilon}$ is an arbitrary constant chosen as small as possible without being zero. Then the recurrence relation

$$\bar{J}_{p-1}(z) = 2pz^{-1}\bar{J}_p(z) - \bar{J}_{p+1}(z) \qquad (50)$$

is used to generate the sequence of functions $\bar{J}_{m-1}(z), \bar{J}_{m-2}(z), \ldots, \bar{J}_1(z), \bar{J}_0(z)$. If m is chosen sufficiently large it can be shown [13] that $J_p(z) = c\bar{J}_p(z)$ for values $0 \leqslant p \leqslant n$. The constant c can be determined with an addition theorem for Bessel functions which, written for $n = 0$, becomes

$$J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) = 1. \qquad (51)$$

In order to generate the Y_n array the zero value is initially obtained from

$$Y_0(z) = \frac{2}{\pi}\left\{J_0(z)\left[\log\frac{z}{2} + \bar{\eta}\right] + 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}J_{2k}(z)}{k}\right\} \qquad (52)$$

using the known values of $J_n(z)$. $Y_1(z)$ follows from

$$J_0(z)\,Y_1(z) = J_1(z)\,Y_0(z) - 2/(\pi z). \qquad (53)$$

The values of $Y_p(z)$ for $p = 2,3 \ldots, N$ are then generated successively from the recurrence relation

$$Y_{p+1}(z) = 2pz^{-1}\,Y_p(z) - Y_{p-1}(z). \qquad (54)$$

This procedure was programmed for the digital computer and used to obtain $J_n(z)$ and $Y_n(z)$ for $|z| \leqslant 10$. The accuracy of the technique was spot-checked by comparing results with recently published tables [11] which contain values of the Kelvin functions for the same range of arguments.

For those cases in which either the ω/α ratio or the index of the term n is so large that $|z| > 10$, the Bessel functions cannot be accurately derived from recurrence techniques. For these large arguments a procedure based on asymptotic series is adopted [13]. $J_n(z)$ and $Y_n(z)$ are calculated for $n = 0,1$ using suitable truncated forms of the asymptotic series. Higher order functions are found by recurrence with equations (50) and (54).

The form of the solution for the case $\omega/\alpha \to \infty$ must be improved by reconsidering the form of the general solution to the differential equation to be in terms of the modified Bessel functions of first and second kinds, and reapplying the boundary conditions. One thus obtains an alternative form for the cylinder coefficients for the same limiting conditions which otherwise produce an asymptotic indeterminacy in the equation for the coefficients.

6. GENERAL RESULTS AND CONCLUSIONS

In concurrence with the results of Olmstead and Raynor [5] who treat the solid cylinder, it has been determined that for an aluminium cylinder, which is either hollow with large wall thickness or solid, that the temperatures in the body vary by only a few degrees if the cylinder is exposed to the radiation environment of interplanetary space. This phenomenon is in sharp contrast to corresponding results for thin-walled cylinders [1, 2]. By using the general hollow cylinder solution to substantiate the validity of the results of [1], it can be concluded that it is the radial conduction terms which provide the discrepancy in the thermal behavior of the corresponding solid and thin-wall cylinders. After the correction of a small numerical error (found by the author) in [1], it was found that for a wall thickness to cylinder diameter ratio of 0·005 that the two solutions agreed at all points on the circumference to within 0·1 degR.

Because physical insight into the phenomena is not well served by the form of the solutions, a series of numerical results has been obtained. In these results the infinite series were truncated after the terms for $|n| = 10$. The thermal property values used in the calculations are typical of a low thermal conductivity material such as natural rubber. The radiation intensity corresponds to the ambient thermal energy density for interplanetary space for solar radiation near the Earth. The cylinder surface was assumed to be a black body so that $a = \varepsilon = 1$. For a corresponding gray body the temperature gradients would be smaller. Using these parameters in equation (39) provides $T_0 = 535{\cdot}03$°R.

In Fig. 2 is shown the variation of surface temperature around the circumference for the full range of rotational speeds. For zero velocity the temperatures are symmetric about the point on the cylinder nearest to the radiant source. Temperature variation is not sinusoidal as for the thin-walled solutions. Rotation destroys the symmetry and is observed to shift the extreme points of surface temperature into the rotational direction. This shifting effect with increasing speed values is observed to reach a maximum condition for an arc of about 37° on the bright side while the minimum temperature shifts through an angle no greater than about 90°. These results compare to equal values of 71° for the shifting arcs for the thin-wall cylinder in [1]. Ölçer [15] has considered the shifting of the extremal temperatures in detail for the solid cylinder problem. The rotation diminishes both maximum and minimum values. For rotational speeds approaching infinity the temperature is everywhere found to approach T_0 as required by thermodynamic equilibrium.

Figure 3 gives results for the identical case of

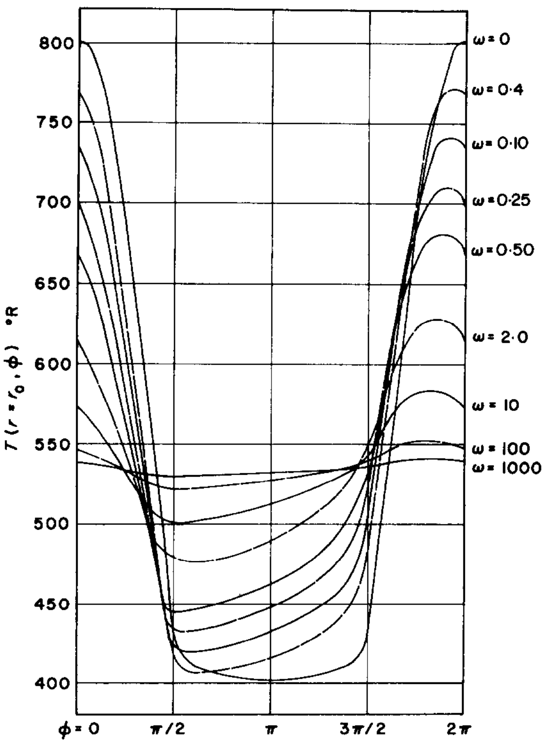

FIG. 2. Outer surface temperature variation for different rotational speeds for the hollow cylinder. ($\psi = \pi/2$ rad, $r_0 = 1$ ft, $r_i = 0{\cdot}5$ ft, $k = 0{\cdot}1$ Btu ft^{-1}h^{-1}R^{-1}, $\alpha = 0{\cdot}004$ ft^{2}h^{-1}, $K_s = 442{\cdot}0$ Btu ft^{-2}h^{-1}.)

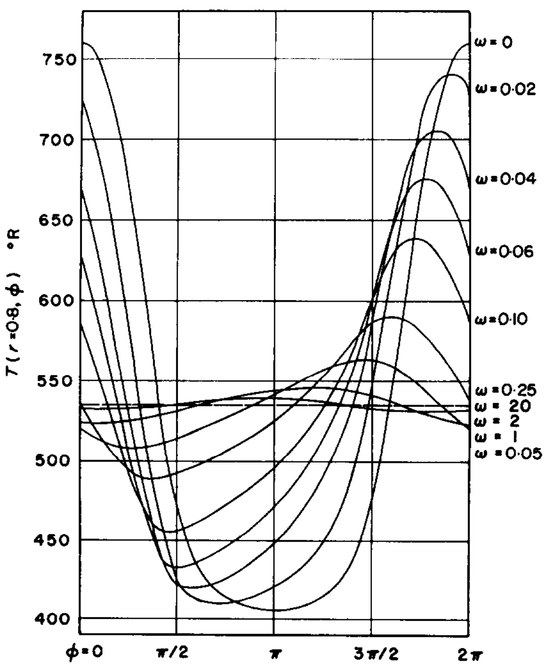

FIG. 3. Hollow cylinder temperatures on the internal surface $\xi = 0{\cdot}8$ as a function of angle ϕ for various rotational speeds. (Same parameters as Fig. 2.)

Fig. 2 but for an internal circumferential surface within the cylinder at $\xi = 0{\cdot}8$. For corresponding values of rotational velocity, on the internal surface there is a greater shifting effect on the extreme temperatures. It is demonstrated by Fig. 3 that the tendency for rotation to damp out the temperature gradients is more pronounced in the interior than near the surface.

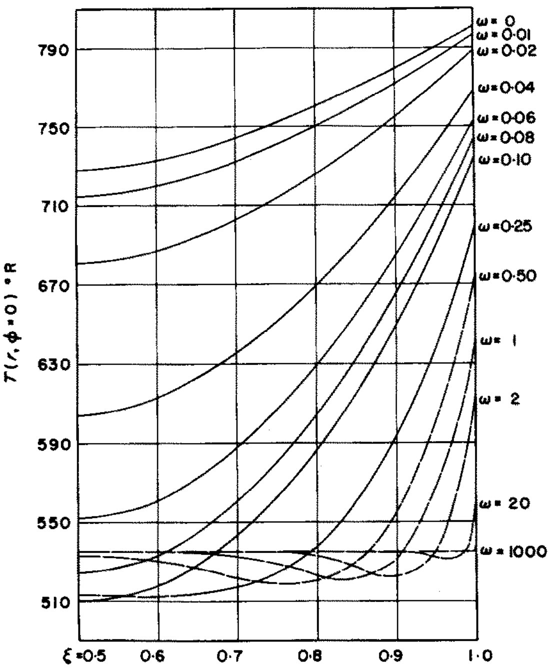

FIG. 4. Temperature as a function of radius in the hollow cylinder for various rotational speeds with the angular position $\phi = 0$. (Same parameters as Fig. 2.)

Figures 4 and 5 give, for two angular positions, respectively, the radial variation of temperature and the dependence of this variation upon rotational speed. These results show that increasing rotation can lead to increased radial gradients. In the aforementioned it was concluded that, in agreement with results of [1] and [2], circumferential gradients always diminish with increasing ω. However, Figs. 4 and 5 indicate that radial temperature gradients in the body can increase with rotation and that

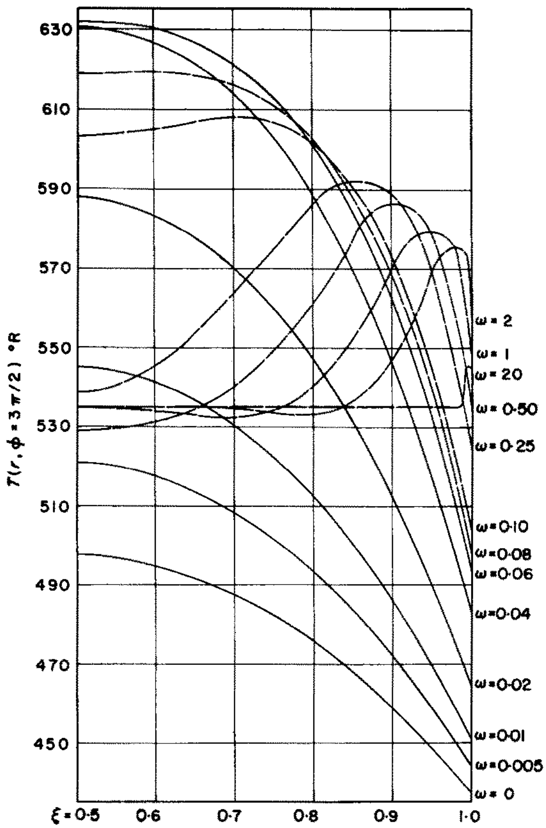

FIG. 5. Hollow cylinder temperature as a function of radius at $\phi = 3\pi/2$ for various values of rotational speed. (Same parameters as Fig. 2.)

larger gradients can be encountered with rotation than would occur for a stationary cylinder.

7. A NUMERICAL SOLUTION FOR THE NONLINEAR RERADIATION FLUX

The method of analysis used in obtaining the general solutions for the solid and hollow cylinders requires that the problem be linear. Therefore it was necessary to linearize the reradiation heat flux term in the external boundary condition which contains the fourth power of temperature. It is of interest to investigate the error in the solution due to this linearization by examining one particular case for which the nonlinear problem can be readily solved.

Raynor and Charnes [1] derived the following nonlinear governing differential equation for the thin-walled cylinder, justifiably omitting terms for radial conduction. Making a minor necessary correction in one of the coefficients as given in the reference, the equation becomes

$$\frac{d^2\tau}{d\varphi^2} + \frac{r^2\omega}{\alpha}\frac{d\tau}{d\varphi} - \frac{r^2\sigma\varepsilon}{sk}\tau^4 = -\frac{r^2aK_s}{sk}\cos^+\varphi \tag{55}$$

where τ is a nonlinear temperature and s is the wall thickness. Anticipating that the maximum linearization error coincides with the greatest temperature fluctuation, it is assumed that the cylinder is stationary. For $\omega = 0$, equation (55) becomes

$$\frac{d^2\tau}{d\varphi^2} - \frac{r^2\sigma\varepsilon}{sk}\tau^4 = -\frac{r^2aK_s}{sk}\cos^+\varphi. \tag{56}$$

The numerical integration of this equation is facilitated by the introduction of the transformation:

$$Y_1 = \tau \qquad Y_2 = \frac{d\tau}{d\varphi}. \tag{57}$$

Then it becomes necessary to solve simultaneously

$$\frac{dY_1}{d\varphi} = Y_2 \tag{58}$$

and

$$\frac{dY_2}{d\varphi} = \frac{r^2}{sk}[\sigma\varepsilon Y_1^4 - aK_s\cos^+\varphi]. \tag{59}$$

This set of first order equations is readily solved with a standard fourth order Runge–Kutta numerical solution technique. It is possible to solve the problem for no rotation as an initial value problem by taking advantage of the symmetry of this solution about $\phi = 0$ to prescribe $[d\tau/d\phi]_{\phi=0} = 0$ and by initiating the calculation with trial values of τ. Then the solution is obtained by an iterative satisfaction of the boundary condition at $\phi = 2\pi$.

Corresponding nonlinear and linear solutions are compared in Fig. 6 for the same numerical example as considered by Charnes and Raynor. It is observed that the linearized temperature solution generally predicts higher temperature than the exact solution. As expected, the error introduced by the approximation is greatest at positions of the surface where the temperature departs the most from the uniform equilibrium value. In Fig. 6, $\hat{T}$ has a

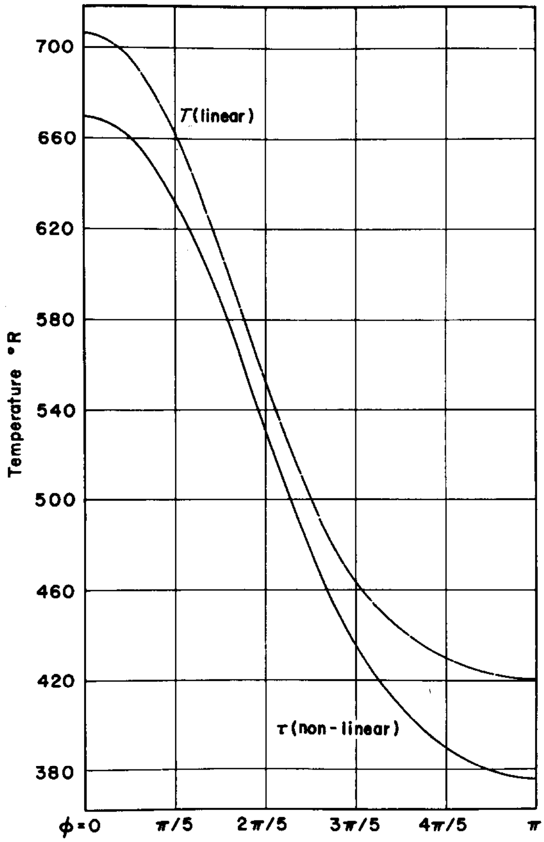

FIG. 6. A comparison of the hollow cylinder outside surface temperatures with the corresponding Runge–Kutta integration of Raynor–Charnes' nonlinear differential equation. ($r_i = 0{\cdot}995$ ft, $k = 100$ Btu ft^{-1} h^{-1} R^{-1}, $\alpha = 3$ ft^2 h^{-1}, $\omega = 0$; other parameters as in above.)

maximum value of 0·22 and the error when compared to the nonlinear solution is about 10 per cent. In the other results the error can be estimated according to the relative value of the associated $\hat{T}$. Moreover, the linear behavior is so very similar to the exact phenomenon that, faced with the inherent difficulty of the non-linear problem, the use of the approximate analytical technique seems justified.

8. COMPARISON OF HOLLOW AND SOLID CYLINDER TEMPERATURES

For the solid and hollow cylinders, Figs. 7 and 8 give radial temperature variation for those radial lines, ϕ = constant, respectively parallel and perpendicular to the direction of incoming radiation. Figure 7 corresponds to no rotation and Fig. 8 depicts the results for $\omega = 0{\cdot}25$. Three different wall thicknesses; $s = 0{\cdot}5$, $s = 0{\cdot}9$ and the solid cylinder ($s = 1{\cdot}0$) are represented.

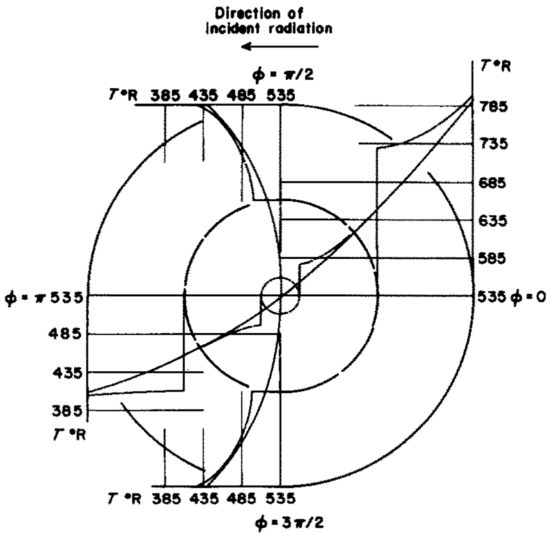

FIG. 7. Radial temperature variation in corresponding solid and hollow cylinders for no rotation. (Solid and hollow $r_o = 1$ ft; hollow $r_i = 0{\cdot}1$ and $0{\cdot}5$ ft; other parameters as in above; $T_0 = 535{\cdot}03$°R.)

The results shown in Fig. 8 which include rotation show that the effect of rotation is to make increasingly less significant the presence of the hole. This observed behavior accrues from the tendency for rotation to damp out temperature variation, particularly on the interior. It can be concluded that the solid cylinder solution is a good approximation to the thick-wall cylinder and that higher rotational speeds extend the valid range of the approximation to walls of lesser thickness.

It is of interest to investigate in greater detail the relationship between cylinder temperatures when a small adiabatic cylindrical hole is

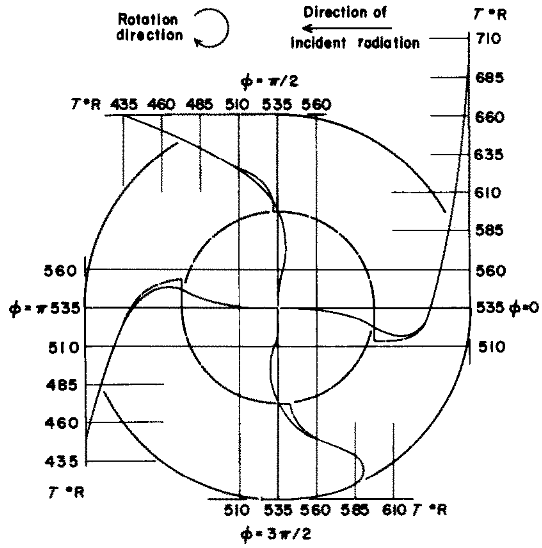

FIG. 8. Radial temperature variation in corresponding solid and hollow cylinders with rotation. (Solid and hollow $r_o = 1$ ft; hollow $r_i = 0{\cdot}5$ ft; $\omega = 0{\cdot}25$ rad/h; other parameters as in above; $T_0 = 535{\cdot}03$°R.)

introduced at the center. One begins by specializing equation (43) for the inner boundary and introducing the Wronskian relation

$$J_n(z)\,Y_n'(z) - J_n'(z)\,Y_n(z) = 2/(\pi z) \qquad (60)$$

and the identity

$$y_p'(z) = y_{p-1}(z) - pz^{-1}\,y_p(z)\;(y = J, Y). \qquad (61)$$

These relations lead to

$$C_n(i^{\frac{1}{2}}n^{\frac{1}{2}}\rho_i) = \frac{\gamma G_n w_i^n}{2^{n-1}n![w_o J_n'(w_o) + \lambda J_n(w_o)]} \qquad (62)$$

for the temperatures around a small hole in the center of the cylinder. For the solid cylinder one obtains for some $r = r_i \ll 1$ the result

$$C_n^{(s)}(i^{\frac{1}{2}}n^{\frac{1}{2}}\rho_i) = \frac{\gamma G_n w_i^n}{2^n n![w_o J_n'(w_o) + \lambda J_n(w_o)]}. \qquad (63)$$

This expression leads to the temperatures on the small circular ring of radius r_i which corresponds to the boundary of the hole considered in equation (62). It follows that

$$\frac{T(r_i, \varphi) - T_0}{T^{(s)}(r_i, \varphi) - T_0} = 2. \tag{64}$$

This result specifies that around the surface of the small adiabatic hole that the temperatures depart from the uniform equilibrium value by twice the value occurring on the corresponding surface in the solid cylinder.

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Résumé—Des solutions analytiques sont obtenues pour la distribution de température dans des cylindres circulaires chauffés par rayonnement à partir d'une source lointaine. On suppose que le cylindre tourne par rapport à la source autour de son axe géométrique en faisant d'un angle constant avec la direction du rayonnement. Les températures quasi-permanentes sont obtenues sous forme de séries de fonctions orthogonales pour le cylindre solide et pour le cylindre creux avec un trou adiabatique.

Les distributions de température qui en résultent sont examinées pour établir l'importance de la rotation, des gradients radiaux de température et de l'épaisseur de la paroi du corps.

Zusammenfassung—Analytische Lösungen liessen sich für die Temperaturverteilung in Kreiszylindern, die von einer entfernt angeordneten Quelle bestrahlt wurden, erhalten. Von dem Zylinder wird angenommen, er rotiere gegenüber der Quelle um seine geometrische Achse und er sei gegen die Strahlungsrichtung um einen konstanten Winkel geneigt. Quasi-stationäre Temperaturen wurden in Form von Reihen aus Orthogonalfunktionsausdrücken für den Vollzylinder und den Hohlzylinder mit adiabater Öffnung gefunden. Die erhaltenen Temperaturverteilungen wurden daraufhin untersucht, ob sie die Charakteristika von Rotation, radialem Temperaturgradienten und Wanddicke des Körpers bestätigen.

Аннотация—Получены аналитические решения для распределения температуры в круглых цилиндрах, нагреваемых излучением от источника, расположенного на большом расстоянии. Предполагается, что цилиндр вращается относительно источника вокруг его геометрической оси, а также, что он имеет наклон под заданным постоянным углом к направлению излучения. Для сплошного цилиндра и для полого цилиндра с адиабатическим отверстием получены квазистационарные решения для температур, представленные в виде разложения в ряд по ортогональным функциям. Полученные распределения температур были проанализированы с точки зрения существенности эффектов вращения, радиальных градиентов температур и толщины стенки тела